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The Chevalley groups $G_2(q)$ with q odd and $2-(v, k, 1)$ designs

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Abstract

Let $\mathcal{D}$ be a $2-(v, k, 1)$ design, and let G be an automorphism group of $\mathcal{D}$. Delandtsheer proved that if $\mathcal{D}$ is not a projective plane and G is line-primitive then G is almost simple, that is, $T \leq G \leq \mathrm{Aut}(T)$, where T is a non-Abelian simple group. In this paper, we prove that T is not isomorphic to $G_2(q)$, where q is odd.

MSC: 05B05; 20B25

Keywords: Line-primitive; Design; Automorphism

1. Introduction

A $2-(v, k, 1)$ design $\mathcal{D} = (\mathbf{P}, \mathbf{L})$ is a system consisting of a finite set $\mathbf{P}$ of v points and a collection $\mathbf{L}$ of k-subsets of $\mathbf{P}$, called lines, such that each 2-subset of $\mathbf{P}$ is contained in precisely one line. We shall always assume that $2 < k < v$.

Let $G \leq \mathrm{Aut}(\mathcal{D})$ be a group of automorphisms of a $2-(v, k, 1)$ design $\mathcal{D}$. The group G is said to be *line-transitive* (*line-primitive*, respectively) on $\mathcal{D}$ if G is transitive (primitive, respectively) on $\mathbf{L}$. The group G is said to be *point-transitive* (*point-primitive*, respectively) on $\mathcal{D}$ if G is transitive (primitive, respectively) on $\mathbf{P}$. A *flag* of $\mathcal{D}$ is a pair consisting of a point and a line containing this point. G is *flag-transitive* on $\mathcal{D}$ if G is transitive on the set of flags of $\mathcal{D}$. The following results are well-known:

(i) if G is line-transitive, then G is also point-transitive (see [2]);
(ii) if G is flag-transitive, then G is point-primitive (see [13]);
(iii) if G acts line-primitively on a finite projective plane, then G is point-primitive (see [14]).

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Delandtsheer [9] conjectured that if G is line-primitive, then G is also point-primitive. Under each of the following hypotheses the conjecture holds true:

(1) $\mathcal{D}$ is a finite projective plane;
(2) $k/(k, v) \leq 10$;
(3) $v > \frac{[k(k-1)/2-1]^2}{2}$;
(4) $k \leq 40$;
(5) G has a subgroup acting regularly on $\mathbf{P}$;
(6) The rank of G acting on $\mathbf{L}$ does not exceed seven;
(7) $\mathrm{Soc}(G) \cong A_n, L_2(q), Sz(2^{2m+1}), {}^2G_2(3^{2m+1}), G_2(2^m)$ or ${}^2F_4(2^{2m+1})$, where $n \geq 5$ and $m \geq 1$.

Here (1), (3), (5) and (6) refer to [9]; (2) and (4) refer to [9, 12, 17] and [20]; (7) refers to [16, 18, 19, 21, 23] and [24]. In this paper, we will prove the following theorem:

Main Theorem. Let $\mathcal{D}$ be a $2-(v, k, 1)$ design and $G \leq \mathrm{Aut}(\mathcal{D})$. If G acts as a line-primitive automorphism group of $\mathcal{D}$, then $\mathrm{Soc}(G)$ is not isomorphic to $G_2(q)$, where q is odd.

2. Some parameters

Let $\mathcal{D}$ be a $2-(v, k, 1)$ design, and let b and v denote the number of the lines and the points of $\mathcal{D}$, respectively. Let r denote the number of the lines containing a given point of $\mathcal{D}$. Then

$$b = \frac{v(v-1)}{k(k-1)}, \qquad r = \frac{(v-1)}{(k-1)}.$$

It is well known that $b \geq v$ and so $k \leq r$. If $k = r$ then $v = k^2 - k + 1$; if $r \geq k+1$, then $v \geq k^2$.

We would like to advertise the following parameters of $2-(v, k, 1)$ designs, introduced in [12]

$$b_1 = (b, v), \qquad b_2 = (b, v-1), \qquad k_1 = (k, v) \qquad \text{and} \qquad k_2 = (k, v-1).$$

Obviously, $k = k_1k_2$, $b = b_1b_2$, $r = b_2k_2$ and $v = b_1k_1$.

Proposition 2.1 ([12]). *Let G be line-transitive and f denote the number of G-orbits on the flags of $\mathcal{D}$, then the following results hold for each flag (α, l):*

(i) $b_1b_2k_1 \mid |(\alpha, l)^G|$; (ii) $k_1 \mid |\alpha^{G_l}|$; (iii) $b_2 \mid |l^{G_\alpha}|$; (iv) $f \leq k_2$.

Corollary 2.1 (Camina–Gagen [4]). *If G is line-transitive and $k \mid v$, then G is flag-transitive.*

The proof follows immediately from Proposition 2.1 (iv), because in this case $k_2 = 1$.

3. Preliminary results

Our conventions for expressing the structure of groups run as follows. If X and Y are arbitrary finite groups, then $X \cdot Y$ denotes an extension of X by Y. The expressions $X : Y$ and $X^{\cdot}Y$ denote split and non-split extensions, respectively. The expression $X \times Y$ denotes the direct product of X and Y, and $X \circ Y$ denotes a central product. The symbol $[m]$ denotes an arbitrary group of order m while Z_m or simply m denotes a cyclic group of that order. Other notation for group structure is standard, except that $SL^{\epsilon}_m(q)$ denotes $SL_m(q)$ or $SU_m(q)$ according to whether ϵ is $+$ or $-$. In addition, we use symbol $p^i \parallel n$ to denote $p^i \mid n$ but $p^{i+1} \nmid n$ and symbol $|n|_p$ to denote the p-part of n, that is, $|n|_p = p^i$ and $p^i \parallel n$, where n and i are positive integers and p is a prime. Let G be a finite group. We use the symbol $|G|$ to denote the order of G and the symbol $|G|_p$ to denote the p-part of $|G|$.

Now let $\mathbf{F} = GF(q)$ be a finite field of order $q = p^e$ (and characteristic p). Let G be the Chevalley group of type G_2 over $\mathbf{F}$. The order of G is

$$q^6(q^2-1)(q^6-1) = q^6(q-1)^2(q+1)^2(q^2+q+1)(q^2-q+1).$$

No prime which is greater than three divides more than one of these factors. Let P_a, P_b be representatives of the two classes of maximal parabolic subgroups of G. Let V and $\mathcal{F}$ be the same as in [1]. We define $\Gamma(V)$ to be the group of all semilinear maps on V. Write $\Gamma(V, \mathcal{F})$ for the subgroup of $\Gamma(V)$ preserving $\mathcal{F}$. Evidently, $\Gamma(V, \mathcal{F})$ is G extended by a field automorphism of order e, and $\mathrm{Aut}(G) = \Gamma(V, \mathcal{F})$ unless $p = 3$. Let $H_1 = \Gamma(V, \mathcal{F})$ and $H_0 = G_2(q)$. Then when $p = 3$, $|\mathrm{Aut}(H_0) : H_1| = 2$, and when $p \geq 5$, $H_1 = \mathrm{Aut}(H_0)$.

Kleidman [15] has determined the maximal subgroups of G.

Lemma 3.1. *Assume that $H_0 \leq H \leq H_1$, where $H_0 \cong G_2(q)$ ($q = p^n$ is odd) and H_1 are as above. Let M be a maximal subgroup of H not containing H_0. Then $M_0 = M \cap H_0$ is H_0-conjugate to one of the following groups:*

Structure	*Order*	*Remarks*
$[q^5] : GL_2(q)$	$q^6(q-1)^2(q+1)$	*Parabolic*
$[q^5] : GL_2(q)$	$q^6(q-1)^2(q+1)$	*Parabolic*
$(SL_2(q) \circ SL_2(q)) \cdot 2$	$q^2(q^2-1)^2$	*Involution centralizer*
$2^{3\cdot}L_3(2)$	$2^6 \cdot 3 \cdot 7$	$q = p$
$SL^{\epsilon}_3(q) : 2$	$2q^3(q^3-\epsilon 1)(q^2-1)$	$\epsilon = \pm$
$G_2(q_0)$	$q_0^6(q_0^2-1)(q_0^6-1)$	$q = q_0^m$, *m prime*
${}^2G_2(q)$	$q^3(q^3+1)(q-1)$	$p = 3$, *n odd*
$PGL_2(q)$	$q(q^2-1)$	$p \geq 7, q \geq 11$
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	$p \geq 5, \mathbf{F} = \mathbf{F}_p[\omega]$ $\omega^3 - 3\omega + 1 = 0$
$L_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	$p \neq 13, \mathbf{F} = \mathbf{F}_p[\sqrt{13}]$
$G_2(2)$	$2^6 \cdot 3^3 \cdot 7$	$q = p \geq 5$
J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	$q = 11,$

Conversely, if $K \leq H_0$ is H_0-conjugate to one of these groups, then $N_H(K)$ is maximal in H.

Lemma 3.2. *Assume that $H_0 < H \leq \mathrm{Aut}(H_0)$, where $H_0 \cong G_2(q)$ $(q = 3^n)$. Further suppose that H contains a graph automorphism of H_0. Let M be a maximal subgroup of H not containing H_0. Then $M_0 = M \cap H_0$ is H_0-conjugate to one of the following groups:*

Structure	*Order*	*Remarks*
$[q^6] : (Z_{q-1})^2$	$q^6(q-1)^2$	*Borel subgroup*
$(SL_2(q) \circ SL_2(q)) : 2$	$q^2(q^2-1)^2$	*Involution centralizer*
$2^{3\cdot} L_3(2)$	$2^6 \cdot 3 \cdot 7$	$q = 3$
$(Z_{q-\epsilon 1})^2 \cdot D_{12}$	$12(q-\epsilon 1)^2$	$q \geq 9, \epsilon = \pm$
$(Z_{q^2+\epsilon q+1}) : Z_6$	$6(q^2+\epsilon q+1)$	$q \geq 9, \epsilon = \pm$
$G_2(q_0)$	$q_0^6(q_0^2-1)(q_0^6-1)$	$q = q_0^m$, *m prime*
${}^2G_2(q)$	$q^3(q^3+1)(q-1)$	*n odd*
$L_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	$q = 3$

Conversely, if $K \leq H_0$ is H_0-conjugate to one of these groups, then $N_H(K)$ is maximal in H.

Lemma 3.3. *$G = G_2(q)$ contains the following subgroups:*

(1) *$T_1 = Z_{q-1} \times Z_{q-1}$ and $N_G(T_1) = T_1 \cdot D_{12}$;*
(2) *$T_2 = Z_{q+1} \times Z_{q+1}$ and $N_G(T_2) = T_2 \cdot D_{12}$;*
(3) *$T_3 = Z_{q^2+q+1}$ and $N_G(T_3) = T_3 : Z_6$;*
(4) *$T_4 = Z_{q^2-q+1}$ and $N_G(T_4) = T_4 : Z_6$;*
(5) *$T_5 = Z_{q^2-1}$ and $N_G(T_5) = T_5 \cdot (Z_2 \times Z_2)$.*

Proof. We can get the above assertions from the information in [11]. $\square$

Definition. Let a, n be two positive integers. A divisor t of $a^n - 1$ is a-primitive, if $t > 0$ and $(t, a^i - 1) = 1, 0 < i < n$.

Lemma 3.4 (Zsigmondy's Theorem [25]). *Let n and a be two positive integers greater than 1. Then $a^n - 1$ possesses an a-primitive divisor, except in the following cases:*

(1) *$n = 2$ and $a = 2^s - 1$, where $s \geq 2$.*
(2) *$n = 6$ and $a = 2$.*

Lemma 3.5. *Let $q = q_0^m$, where m is an odd prime, and let ϵ be $\pm$. Then*

(1) *$(q - \epsilon 1)^2$ does not divide $|G_2(q_0)|$;*
(2) *$(q^2 + \epsilon q + 1)$ does not divide $|G_2(q_0)|$.*

Proof. Note that

$$|G_2(q_0)| = q_0^6(q_0^2 - q_0 + 1)(q_0^2 + q_0 + 1)(q_0^2 - 1)^2$$

and $(q-\epsilon 1)^2 = (q_0^m - \epsilon 1)^2$. So both $(q_0^3 - \epsilon 1)^2$ and $(q_0^5 - \epsilon 1)^2$ do not divide $|G_2(q_0)|$. If $m \geq 7$, then by Lemma 3.4, there exists a prime t, such that t divides $q_0^m - \epsilon 1$ but $t \nmid (q_0^6 - 1)$, and so (1) holds. Hence we deduce that assertion (1) is true.

By Lemma 3.4, there exists a q-primitive divisor t of $q^6 - 1$. Then $t \mid (q^2 - q + 1)$ but $t \nmid (q_0^6 - 1)$. Hence $q^2 - q + 1$ does not divide $|G_2(q_0)|$. If $m \neq 2$, then by Lemma 3.4, $q^3 - 1 = q_0^{3m} - 1$ always possesses a q_0-primitive divisor t. Then t divides $q^2 + q + 1$ but does not divide $q_0^6 - 1$, and so $q^2 + q + 1$ does not divide $|G_2(q_0)|$. Hence assertion (2) is true. □

Lemma 3.6. *Let $\mathcal{D}$ be a $2-(v,k,1)$ design, and let G be a line-transitive automorphism group of $\mathcal{D}$. Assume that K is a subgroup of G and $K \not\leq G_L$ for any $L \in \mathbf{L}$. If $K \leq G_\alpha$ for some $\alpha \in \mathbf{P}$, then $N_G(K) \leq G_\alpha$.*

Proof. If $N_G(K) \not\leq G_\alpha$, then there exists $g \in N_G(K)$, such that $\alpha^g \neq \alpha$. Set $\alpha^g = \beta$. Let L be the line containing α and β. Since $\beta^K = \alpha^{gK} = \alpha^{Kg} = \alpha^g = \beta$ and $K \leq G_\alpha$, we have $K \leq G_L$, which conflicts with our hypothesis. Hence $N_G(K) \leq G_\alpha$. □

Lemma 3.7. *Let $\mathcal{D}$ be a $2-(v,k,1)$ design, and let $G \leq \mathrm{Aut}(\mathcal{D})$ be line-transitive. If $\mathcal{D}$ is not a finite projective plane, then, for any prime p dividing b_2, there exists a Sylow p-subgroup P of G, such that $N_G(P) \leq G_\alpha$ for some $\alpha \in \mathbf{P}$.*

Proof. Since $\mathcal{D}$ is not a projective plane, $b > v$ and $b_2 > 1$. Thus there exists at least one prime p dividing b_2. Let P be a Sylow p-subgroup of G. Then P does not fix any line L of $\mathbf{L}$. Since b_2 divides $v-1$, p does not divide v. Hence P must fix some point α of $\mathbf{P}$. By Lemma 3.6, the conclusion holds. □

Lemma 3.8. *Let $\mathcal{D}$ be a $2-(v,k,1)$ design, and let $G \leq \mathrm{Aut}(\mathcal{D})$ be line-transitive. Assume that P is a Sylow p-subgroup of G_α for some α of $\mathbf{P}$. If P is not a Sylow p-subgroup of G, then there exists a line L through α such that $P \leq G_L$.*

Proof. Since P is not a Sylow subgroup of G, there exists a Sylow subgroup Q of G, such that $P < Q$. Thus $N_Q(P) > P$. For any $g \in N_Q(P)\backslash P$, $g \notin G_\alpha$ since P is a Sylow p-subgroup of G_α. By Lemma 3.6, $P \leq G_L$ for some $L \in \mathbf{L}$. □

Assume that G is a line-primitive but point-imprimitive automorphism group of $\mathcal{D}$. Let

$$\mathcal{C} = \{C_1, C_2, \ldots, C_c\}$$

be a non-trivial partition of $\mathbf{P}$ into c classes of size s, which is preserved by G and G acts primitively on it. By [10], G acting on $\mathcal{C}$ is faithful. Note however that Lemma 3.9 below holds also if the line-primitivity hypothesis is weakened into line-transitivity.

Lemma 3.9 ([12]). *There exist positive integers x and y such that $s = xb_2 + 1$ and $c = yb_2 + 1$.*

Lemma 3.10 (Lemma 4 of [5]). *Let G be a line-transitive automorphism group of* $2-(v,k,1)$ *design. If there is an involution of G which has no fixed point, then* $k \mid v$, *and so G is flag-transitive.*

Lemma 3.11. *Let G be a transitive group on* Ω, *and K be a conjugate class of an element of G. Let* $x \in K$ *and* $\mathrm{Fix}(\langle x\rangle)$ *denote the set of points fixed by* $\langle x\rangle$. *Then*

$$|\mathrm{Fix}(\langle x\rangle)| = |G_\alpha \cap K| \cdot |\Omega|/|K|,$$

where $\alpha \in \Omega$.

Proof. Count the number of the order pairs (α, x), where $\alpha \in \mathrm{Fix}(\langle x\rangle)$. □

Lemma 3.12. *Let G be a line-transitive automorphism group of* $2-(v,k,1)$ *design* $\mathcal{D}$. *Suppose that* $G \cong G_2(q)$, *where q is odd. If* G_L *has a unique conjugucy class of involutions, then* $G_\alpha \ncong Z_{q^2+\epsilon q+1} : Z_6$, *where* $\epsilon = \pm$, *L is a line of* $\mathcal{D}$ *and* α *is a point of* $\mathcal{D}$.

Proof. Suppose that $G_\alpha \cong Z_{q^2+\epsilon q+1} : Z_6$. Let i be an involution of G. Let $H = \langle i\rangle$. If i has no fixed point, then by Lemma 3.11, G is flag-transitive. By the main theorem of [3], this is impossible. Thus i has at least two fixed points (note that here v is even). By Lemma 2 of [5], we get either $\mathrm{Fix}(H) \subseteq L$ or the induced structure on $\mathrm{Fix}(H)$ is a $2-(v_0,k_0,1)$ design $\mathcal{D}_0$ and $N_G(H)$ acts line-transitively on this design, where $v_0 = |\mathrm{Fix}(H)|$, $k_0 = |\mathrm{Fix}(H) \cap L|$. By Lemma 3.6 of [21], we know that only the latter occurs. Note that $G_2(q)$ has a unique conjugucy class of involutions (see [7]). We use $e(G)$ to denote the number of involutions of G. Then by Lemma 3.10, we get

$$|\mathrm{Fix}(H)| = |G : G_\alpha| \cdot e(G_\alpha)/e(G) = |C_G(i)| \cdot e(G_\alpha)/|G_\alpha| = q^2(q^2-1)^2/6.$$

Hence $v_0 = q^2(q^2-1)^2/6$. Let b_0 denote the number of lines of $\mathcal{D}_0$. Then b_0 divides $|N_G(H)|$, and so $b_0 = q^2(q^2-1)^2/a$, where $1 \le a \le 6$ (note that here $b_0 \ge v_0$). From $b_0k_0(k_0-1) = v_0(v_0-1)$, we get $6k_0(k_0-1) = a(v_0-1)$. Since v_0 is even, $a = 4$. Thus we have $3 \mid (v_0-1)$, which conflicts with the fact that $3 \mid v_0$. This contradiction shows that $G_\alpha \ncong Z_{q^2+\epsilon q+1} : Z_6$. □

The following lemma is a generalization of Lemma 2.2 of [6].

Lemma 3.13. *Let G be a group acting line-transitively on a* $2-(v,k,1)$ *design* $\mathcal{D}$. *Let g be an element of order s of* G_L, *where s is a prime and L is a line of* $\mathcal{D}$. *Assume that there is a normal subgroup N of G with* $|G:N| = s$, *such that* $g \notin N$. *Then also N acts line-transitively.*

Proof. Since $N \trianglelefteq G$, $N \cap G_L = N_L \trianglelefteq G_L$. By $g \in G_L$ and $g \notin N$, we get $N < NG_L \le G$. Because $|G : N| = s$ is a prime, thus $G = NG_L$. Hence

$$G_L/N_L = G_L/(N \cap G_L) \cong NG_L/N = G/N,$$

and so $|G/G_L| = |N/N_L|$. It follows that N is line-transitive. □

4. The proof of the Main Theorem

For proving the Main Theorem, we first prove the following two propositions.

Proposition 4.1. *Let $\mathcal{D}$ be a $2-(v,k,1)$ design, and let G be a group of automorphisms of $\mathcal{D}$, and suppose that $H_0 = G_2(3^n) \trianglelefteq G \leq \mathrm{Aut}(G_2(3^n))$ with $n \geq 1$. Further suppose that G contains a graph automorphism of H_0. If G is line-primitive, then G is also point-primitive.*

Proof. Let $q = 3^n$. Since $G_2(q) \trianglelefteq G$ and G is line-primitive, $G_2(q)$ is line-transitive. For any $L \in \mathbf{L}$, G_L is a maximal subgroup of G by Theorem 8.2 of [22]. Since $G_2(q)$ is line-transitive, $G_2(q) \not\leq G_L$ for any $L \in \mathbf{L}$. Hence $G_L \cap G_2(q)$ is a group occurring in Lemma 3.2. Suppose that G is a point-imprimitive automorphism group of $\mathcal{D}$. Then $\mathcal{D}$ is not a projective plane. Therefore $b > v$. Let

$$\mathcal{C} = \{C_1, C_2, \ldots, C_c\}$$

be a non-trivial partition of $\mathbf{P}$ into c classes of size s, which is preserved by G and G acts primitively on it. By [10], G acting on $\mathcal{C}$ is faithful. For any imprimitivity class $C \in \mathcal{C}$, we know that G_C is a maximal subgroup of G. Thus $G_C \cap H_0$ is a group occurring in Lemma 3.2. Further, $|G_L| < |G_C|$. Now we know that $b = |H_0|/|H_0 \cap G_L|$ for any line L and $v = |H_0|/|H_0 \cap G_\alpha|$ for any $\alpha \in \mathbf{P}$. Of course, $|G_L \cap H_0| < |G_\alpha \cap H_0|$, and so $G_\alpha \cap H_0 \not\leq G_L \cap H_0$. By Lemma 3.6, we deduce that $N_G(G_\alpha \cap H_0) \leq G_\alpha$. On the other hand, by Lemma 3.7, there exists a prime t, such that $t \mid b$ and G_α contains a Sylow t-subgroup T of G and $N_G(T) \leq G_\alpha$ for some $\alpha \in \mathbf{P}$. If $t \geq 3$, then by Lemma 3.3, we know that $N_G(T)$ contains B_0 or $(Z_{q-\epsilon 1})^2 \cdot D_{12}$ or $Z_{q^2+\epsilon q+1} : Z_6$, where B_0 denotes a Borel subgroup of H_0. Hence $G_\alpha \cap H_0$ is one of the three groups. Hence by Lemma 3.2, we know that G_α is a maximal subgroup of G, and so G acts primitively on $\mathbf{P}$, a contradiction. If $t = 2$, then by Lemma 3.7, we can assume that $b_2 = 2^i$, where $i \geq 1$. Let $2^a \parallel (q-\epsilon 1)$, $a \geq 2$. Then $T \cong (Z_{2^a} \times Z_{2^a}) \cdot (Z_2 \times Z_2)$ by Lemma 3.3. If $Z_{2^a} \times Z_{2^a} \not\leq G_L$, then by Lemma 3.6, $N_G(Z_{2^a} \times Z_{2^a}) \leq G_\alpha$. Since

$$Z_{2^a} \times Z_{2^a} \quad \text{char} \quad Z_{q-\epsilon 1} \times Z_{q-\epsilon 1} \trianglelefteq (Z_{q-\epsilon 1} \times Z_{q-\epsilon 1}) \cdot D_{12},$$

we have $(Z_{q-\epsilon 1} \times Z_{q-\epsilon 1}) \cdot D_{12} \leq G_\alpha$. Note that $(Z_{q-\epsilon 1} \times Z_{q-\epsilon 1}) \cdot D_{12} \not\leq G_L$. By Lemma 3.6, $N_G(Z_{q-\epsilon 1} \times Z_{q-\epsilon 1} \cdot D_{12}) \leq G_\alpha$. Again by Lemma 3.2 we deduce that G_α is a maximal subgroup of G. Therefore $Z_{2^a} \times Z_{2^a} \leq G_L$. Since $2 \mid b$, by Lemma 3.2, we know that $G_L \cap H_0 \cong [q^6] : (Z_{q-1})^2$. This conflicts with $|G_L \cap H_0| < |G_\alpha \cap H_0|$. This completes the proof of Proposition 4.1. □

Proposition 4.2. *Let $\mathcal{D}$ be a $2-(v,k,1)$ design, and let G be a group of automorphisms of $\mathcal{D}$, and suppose that $G_2(p^n) \trianglelefteq G \leq H_1$ with p is odd and $n \geq 1$, and H_1 is as above. If G is line-primitive, then G is also point-primitive.*

Proof. Let $q = p^n$. Since $G_2(q) \trianglelefteq G$ and since G is line-primitive, $G_2(q)$ is line-transitive. For any $L \in \mathbf{L}$, G_L is a maximal subgroup of G by Theorem 8.2 of [22]. By Lemma 3.1, $G_L \cap G_2(q)$ is a maximal subgroup of $G_2(q)$ or $G_2(q) \leq G_L$. Since $G_2(q)$ is line-transitive, $G_2(q) \not\leq G_L$ for any $L \in \mathbf{L}$. Thus $G_2(q)$ acts primitively on $\mathbf{L}$ by Theorem 8.2 of [22], and so we may suppose that $G = G_2(q)$.

Suppose that G is a point-imprimitive automorphism group of $\mathcal{D}$. Let

$$\mathcal{C} = \{C_1, C_2, \ldots, C_c\}$$

be a non-trivial partition of $\mathbf{P}$ into c classes of size s, which is preserved by G and G acts primitively on it. Since G is a simple group, G acting on $\mathcal{C}$ is faithful. For any imprimitivity class $C \in \mathcal{C}$, we know that G_C is a maximal subgroup of G. Thus G_L and G_C are groups occurring in the statements of Lemma 3.1. Further, $|G_L| < |G_C|$.

By (iii) occurring in the Introduction, we can assume that $\mathcal{D}$ is not a projective plane. Thus by Lemma 3.7, for any prime t dividing b_2, there exists a Sylow t-subgroup T of G, such that $N_G(T) \leq G_\alpha$ for some $\alpha \in \mathbf{P}$. Let C be an imprimitivity class of G and $\alpha \in C$. Then $G_\alpha < G_C$.

Note that $b_2 \mid |G|$, that is, b_2 divides $q^6(q^2-1)^2(q^2+q+1)(q^2-q+1)$. Hence we can prove this proposition in several steps that we give as Lemmas. This is the first one.

Lemma 4.1. *p does not divide b_2.*

Proof. Suppose that $p \mid b_2$. Then we take that $t = p$. Thus $N_G(T)$ is a Borel subgroup of G. Thus $G_C = P_a$ or P_b and so by Lemma 3.7 $G_\alpha = N_G(P)$. It follows that $|G_\alpha| = |N_G(P)| = q^6(q-1)^2$. Since $v = |G/G_\alpha| = (q+1)(q^3+1)(q^2+q+1)$, $v - 1 = q^6+2q^5+2q^4+2q^3+2q^2+2q$. Thus $|v-1|_p = q$. It follows that $|b|_p \leq q$ and so $q^5 \mid |G_L|$. This forces that G_L is isomorphic to P_a or P_b by Lemma 3.1, which conflicts with $|G_L| < |G_\alpha|$. $\square$

Lemma 4.2. *If t divides $(b_2, q-\epsilon 1)$, then $t < 5$, where $\epsilon = \pm$.*

Proof. Suppose that $t \geq 5$. Then by Lemmas 3.3 and 3.7 we have $(Z_{q-\epsilon 1})^2 \cdot D_{12} \leq N_G(T) \leq G_\alpha$. Hence $G_C \cong SL_3^\epsilon(q) : 2$ and $c = q^3(q^3+\epsilon 1)/2$ and $c - 1 = (q^3-\epsilon 1)(q^3+\epsilon 2)/2$. Since $(Z_{q-\epsilon 1})^2 \cdot D_{12}$ is maximal in the group $SL_3^\epsilon(q) : 2$ and $G_\alpha \leq SL_3^\epsilon(q) : 2$, we have $G_\alpha = (Z_{q-\epsilon 1})^2 \cdot D_{12}$ and $v = q^6(q+\epsilon 1)^2(q^4+q^2+1)/12$. Since $|G_L| < |G_\alpha|$, we have $G_L \cong 2^{3\cdot}L_3(2)$ or $L_2(8)$ or $L_2(13)$ or $G_2(2)$ or $PGL_2(11)$ $(q = 11)$ or $G_2(q_0)$, where $q = q_0^m$ and m is a prime with $m \geq 7$. $\square$

If $G_L \cong 2^{3\cdot}L_3(2)$, then $|G_L| = 2^6 \cdot 3 \cdot 7$ and $q = p$. Assume that 7 does not divide $q - \epsilon 1$. Then $t \neq 7$ and $t^2 \mid |G_\alpha|$. Since $bk = vr$ and G is line-transitive, we have

$$k_1|G_\alpha| = b_2|G_L|. \tag{1}$$

Thus $t^2 \mid b_2$, which conflicts with the fact that $b_2 \mid (c-1)$ (see Lemma 3.9). Therefore, we can suppose that $7 \mid (q-\epsilon 1)$. If $2 \parallel (q+\epsilon 1)$, then v is odd, and so k_1 is odd, too. By (1), we get $b_2 = (q-\epsilon 1)^2/(16 \cdot 7)$. Since $b_2 \mid (c-1)$, we have $q - \epsilon 1$ divides $56(q^2+\epsilon q+1)(q^3+\epsilon 2)$. Note that

$$(q-\epsilon 1, q^2+\epsilon q+1) = 1 \text{ or } 3 \qquad \text{and} \qquad (q-\epsilon 1, q^2+\epsilon 2) = 1 \text{ or } 3.$$

Thus we get $q - \epsilon 1$ divides $2^3 \cdot 3^2 \cdot 7$. Note that 28 divides $q - \epsilon 1$, and so we have

$$q \in \{29, 83, 167, 251, 503\}.$$

If $2 \parallel (q - \epsilon 1)$, then v is even, and so by (1) we have $b_2 = (q - \epsilon 1)^2/28$. Similarly, we can deduce that $(q - \epsilon 1)$ divides $2 \cdot 3^2 \cdot 7$. Hence

$$q \in \{13, 41, 43, 127\}.$$

Recall that

$$bk(k-1) = v(v-1).$$

It follows that

$$k(k-1) = \frac{|G_L|(v-1)}{|G_\alpha|}. \tag{2}$$

Hence when $G_\alpha = (Z_{q-\epsilon 1})^2 \cdot D_{12}$ and $G_L = 2^{3\cdot} L_3(2)$, Eq. (2) is replaced by the equation

$$k(k-1) = \frac{28(q^6(q+\epsilon 1)^2(q^4+q^2+1) - 12)}{3(q-\epsilon 1)^2}. \tag{3}$$

When q is one of the above values, Eq. (3) has no integer number solutions for k. Thus $G_L \ncong 2^{3\cdot} L_3(2)$.

If $G_L \cong G_2(2)$, then $q = p \geq 5$ and

$$b = q^6 \cdot \frac{q^4+q^2+1}{3} \cdot \left(\frac{q+\epsilon 1}{2}\right)^2 \cdot \left(\frac{q-\epsilon 1}{2}\right)^2 \Big/ (2^2 \cdot 3^2 \cdot 7).$$

Similarly, we can assume that $7 \mid (q - \epsilon 1)$. Thus by (1) we get $b_2 = (q - \epsilon 1)^2/(28a)$, where $a \mid 36$. Further if a is even, then $4 \mid a$; if $3 \mid a$, then $9 \mid a$. Thus by Lemma 3.9, we have $b_2 \mid (c-1)$. This deduces $(q - \epsilon 1)/(14a)$ divides 9, and so

$$q \in \{13, 41, 43, 127, 167, 379, 503, 1511\}.$$

Now Eq. (2) is replaced by the equation

$$k(k-1) = \frac{84(q^6(q+\epsilon 1)^2(q^4+q^2+1) - 12)}{(q-\epsilon 1)^2}. \tag{4}$$

When q is one of the above values, Eq. (4) has no integer number solutions for k. Thus $G_L \ncong G_2(2)$.

If $G_L \cong G_2(q_0)$, where $q = q_0^m$ and $m \geq 7$ a prime, then $b_2 = (q - \epsilon 1)^2/(q_0 - \epsilon 1)^2$, which conflicts with Lemma 3.9.

Now suppose that $G_L \in \{PGL_2(11), L_2(8), L_2(13)\}$. Let $2^i \parallel (q - \epsilon 1)$, then the 2-part of $|G_\alpha|$ is 2^{2i+2}. But $|PGL_2(11)|_2 = 8$, $|L_2(8)|_2 = 8$ and $|L_2(13)|_2 = 4$, hence by Lemma 3.8, $i \geq 2$ and v odd. Therefore the 2-part of b is 8 or 2^{2i-1} or 2^{2i} (recall that when $G_L \cong PGL_2(11)$, $q = 11$). However, each of the above cases conflicts with the fact that $b_2 \mid (c-1)$ since $2^{i-1} \parallel (c-1)$. Therefore Lemma 4.2 is proved.

Lemma 4.3. *If t divides $(b_2, q^2 + \epsilon q + 1)$, then $t < 5$, where $\epsilon = \pm$.*

Proof. Suppose that $t \geq 5$. Then by Lemmas 3.3 and 3.7, we get $Z_{q^2+\epsilon q+1} : Z_6 \leq N_G(T)$. Thus $G_C \cong SL_3^{\epsilon}(q) : 2$ for some imprimitivity class C. Let $\alpha \in C$, then $G_\alpha \cong Z_{q^2+\epsilon q+1} : Z_6$ as before. By $|G_L| < |G_\alpha|$, we get

$$G_L \in \{2^{3\cdot}L_3(2), L_2(8), L_2(13), G_2(2), G_2(q_0)\}.$$

Since each of the groups $L_2(8)$, $L_2(13)$ and $G_2(q_0)$ has a unique conjugacy class of involutions, by Lemma 3.12, none of the three groups is isomorphic to G_L. If $G_L \cong 2^{3\cdot}L_3(2)$, then by Table III of [15], G_L has a unique conjugacy class of elements of order 3. Since $|G_L|_3 = 3$, $3 \nmid (q^2 + \epsilon q + 1)$ by Lemma 3.8. Let $l(G)$ denote the number of elements of order 3 of G. Then $l(G_\alpha) = l(Z_{q^2+\epsilon q+1} : 3) = 2(q^2 + \epsilon q + 1)$. Let h be an element of order 3 of G_α and K be a G-conjugate class of h. Then by Lemma 3.10, $|\mathrm{Fix}(\langle h\rangle)| = |C_G(h)| \cdot |G_\alpha \cap K|/|G_\alpha|$. When h is central in a Sylow 3-subgroup of G, then $C_G(h) \cong SL_3^{-\epsilon}(q)$ and $G_\alpha \cap K = l(G_\alpha)$. Thus $|\mathrm{Fix}(\langle h\rangle)| = |SL_3^{-\epsilon}(q)|/3$. Let $H = \langle h\rangle$. Then H satisfies the condition in Lemma 2 of [5]. Hence $\mathrm{Fix}(H) \subseteq L$ or the induced structure on $\mathrm{Fix}(H)$ is a $2-(v_0, k_0, 1)$ design, where $v_0 = |\mathrm{Fix}(H)|$, $k_0 = |\mathrm{Fix}(H) \cap L|$. Further $N_G(H)$ is line-transitive in this design. Since $v \geq k^2$, $\mathrm{Fix}(H) \nsubseteq L$. Therefore there exists a $2-(v_0, k_0, 1)$ design. Let b_0 denote the number of lines of this design. Then b_0 divides $|N_G(H)|/3$ (note that every point of $\mathrm{Fix}(H)$ is fixed by H). It follows that $b_0 = v_0$ or $2v_0$. This conflicts with $b_0k_0(k_0 - 1) = v_0(v_0 - 1)$ (note that here v_0 is even). When h is not central in any Sylow 3-subgroups of G, then $|C_G(h)| = q(q + \epsilon 1)^2(q - \epsilon 1)$ (see [7]) and $|G_\alpha \cap K| = 2(q^2 + \epsilon q + 1)$. Hence $|\mathrm{Fix}(H)| = q(q + \epsilon 1)^2(q - \epsilon 1)/3$. As before either $\mathrm{Fix}(H) \subseteq L$ or the induced structure on $\mathrm{Fix}(H)$ is a $2-(v_0, k_0, 1)$ design, where $v_0 = |\mathrm{Fix}(H)|, k_0 = |\mathrm{Fix}(H) \cap L|$. Further $N_G(H)$ is line-transitive in this design. If $\mathrm{Fix}(H) \subseteq L$, then $3 \mid k$. (Indeed, since $|G_\alpha| > |G_L|$, we have $q \neq 3$. Consider the cycle decomposition of h acting on $\mathbf{P}$, we have $k = v_0 + 3k'$, where k' is a positive integer. Since $v_0 = q(q+\epsilon 1)^2(q-\epsilon 1)/3$ and $3 \nmid (q-\epsilon 1)$ and $q \neq 3$, we have $3 \mid (q+\epsilon 1)$. This deduces that $3 \mid v_0$, and so $3 \mid k$.) Since $bk(k-1) = v(v-1)$, we have $k(k-1) = |G_L|(v-1)/|G_\alpha|$, that is, $k(k - 1) = 2^5 \cdot 7 \cdot (v - 1)/(q^2 + \epsilon q + 1)$, which is impossible since $3 \mid v$ and $3 \mid k$. Therefore we obtain a $2-(v_0, k_0, 1)$ design. Let b_0 denote the number of lines of this design. Then b_0 divides $|N_G(H)|/3$ (note that every point of $\mathrm{Fix}(H)$ is fixed by H). It follows that $b_0 = v_0$ or $2v_0$. This conflicts with $b_0k_0(k_0 - 1) = v_0(v_0 - 1)$ (note that here v_0 is even).

If $G_L \cong G_2(2)$, then G_L has two conjugate classes of elements of order 3 by [8]. Further, if two elements of order 3 of G_L are conjugate in G, then they are conjugate in G_L. Let h be an element of order 3 which is central in a Sylow 3-subgroup of G. Then by Lemma 3.10, $|\mathrm{Fix}(\langle h\rangle)| = |C_G(h)| \cdot |G_\alpha \cap K|/|G_\alpha|$, where K denotes a conjugacy class of $\langle h\rangle$ in G. If $3 \mid (q - \epsilon 1)$, then $G_C = N_G(\langle h\rangle)$ and $C \subseteq \mathrm{Fix}(\langle h\rangle)$. Indeed, for any point $\beta \in C$, there exists $g \in G_C$, such that $\beta = \alpha^g$. Hence $\beta^{\langle h\rangle} = \alpha^{g\langle h\rangle} = \alpha^{\langle h\rangle g} = \alpha^g = \beta$. This implies that $C \subseteq \mathrm{Fix}(\langle h\rangle)$. Again note that $|\mathrm{Fix}(\langle h\rangle)| = q^3(q - \epsilon 1)(q^2 - 1)/3 = |G_C|/|G_\alpha| = |C|$, and so $C = \mathrm{Fix}(\langle h\rangle)$. This deduces that $\mathrm{Fix}(\langle h\rangle) \nsubseteq L$ since $b > v > s$. Therefore, by Lemma 2 of [5], we obtain a $2-(v_0, k_0, 1)$ design, where $v_0 = |\mathrm{Fix}(\langle h\rangle)|$, $k_0 = |\mathrm{Fix}(\langle h\rangle) \cap L|$ and $N_G(\langle h\rangle)$ acts line-transitively on this design. Let $N = SL_3^{\epsilon}(q)$. Then from Lemma 3.13, N acts line-transitively and, consequently, point-transitively. Hence $|N_\alpha| = |N|/|\mathrm{Fix}(\langle h\rangle)| = 3(q^2 + \epsilon q + 1)$, an odd number. This means that any involution of N has no fixed point. By Lemma 3.11, N is flag-transitive, which contradicts with the main theorem of [3]. If $3 \mid (q+\epsilon 1)$, then $C_G(h) \cong SL_3^{-\epsilon}(q) : 2$. In this case, $|\mathrm{Fix}(\langle h\rangle)| = q^3(q^3+\epsilon 1)(q^2-1)/3$. Since $v \geq k^2$, $\mathrm{Fix}(\langle h\rangle) \not\subseteq L$. Hence by Lemma 2 of [5], again we obtain a $2-(v_0, k_0, 1)$ design, where $v_0 = |\mathrm{Fix}(\langle h\rangle)|$, $k_0 = |\mathrm{Fix}(\langle h\rangle) \cap L|$ and $N_G(\langle h\rangle)$ acts line-transitively on this design. Let $N = SL_3^{-\epsilon}(q)$. Again by Lemma 3.13, we deduce that N is line-transitive. We use b_0 to denote the number of lines of the above design. Then b_0 divides $|N|/3$ (note that here every point of $\mathrm{Fix}(\langle h\rangle)$ is fixed by $\langle h\rangle$). Since $b_0 \geq v_0$, we have $b_0 = v_0 = q^3(q^3+\epsilon 1)(q^2-1)/3$. Therefore, this design is a projective plane, and so v_0 is odd, a contradiction. This last contradiction showed that Lemma 4.3 is true. □

Lemma 4.4. *b_2 is odd.*

Proof. Suppose that b_2 is even. Then by Lemmas 4.1–4.3, we can assume that $b_2 = 2^i \cdot 3^j$, where $i \geq 1$ and $j \geq 0$. Let $2^a \parallel (q-\epsilon 1)$ with $a \geq 2$. Then $T \cong (Z_{2^a} \times Z_{2^a}) \cdot (Z_2 \times Z_2)$. Suppose that $Z_{2^a} \times Z_{2^a} \leq G_L$, then by $|G_L| < |G_\alpha|$ and $2 \mid b$ and Lemma 3.1, there is no such G_L. Hence $Z_{2^a} \times Z_{2^a} \not\leq G_L$, and so $N_G(Z_{2^a} \times Z_{2^a}) \leq G_\alpha$ from Lemma 3.6. Since

$$Z_{2^a} \times Z_{2^a} \quad \text{char} \quad Z_{q-\epsilon 1} \times Z_{q-\epsilon 1} \trianglelefteq (Z_{q-\epsilon 1} \times Z_{q-\epsilon 1}) \cdot D_{12},$$

we have $(Z_{q-\epsilon 1} \times Z_{q-\epsilon 1}) \cdot D_{12} \leq G_\alpha$. In this case, $G_C \cong SL_3^\epsilon(q) : 2$ for some imprimitivity class C. By the maximality of $(Z_{q-\epsilon 1} \times Z_{q-\epsilon 1}) \cdot D_{12}$ in G_C, we have $G_\alpha \cong (Z_{q-\epsilon 1} \times Z_{q-\epsilon 1}) \cdot D_{12}$ and $v = q^6(q^4+q^2+1)(q+\epsilon 1)^2/12$. Since $|G_L| < |G_\alpha|$ and $Z_{2^a} \times Z_{2^a} \not\leq G_L$, we get $G_L \cong 2^{3\cdot} L_3(2)$ or $G_2(2)$.

If $G_L \cong 2^{3\cdot} L_3(2)$, then $q = p$ and

$$b = q^6 \cdot \frac{q^4+q^2+1}{3} \cdot \left(\frac{q+\epsilon 1}{2}\right)^2 \cdot \left(\frac{q-\epsilon 1}{2}\right)^2 \Big/ (2^2 \cdot 7).$$

v is odd. Thus by (1) we have $b_2 = (q-\epsilon 1)^2/16$. Similarly, we get that $(q-\epsilon 1)$ divides 72. By $2 \mid b_2$, we know that $8 \mid (q-\epsilon 1)$. Therefore, we deduce that

$$q \in \{7, 23, 71, 73\}.$$

We can obtain a contradiction as in Lemma 4.2.

If $G_L \cong G_2(2)$, then $q = p$ and

$$b = q^6 \cdot \frac{q^4+q^2+1}{3} \cdot \left(\frac{q+\epsilon 1}{2}\right)^2 \cdot \left(\frac{q-\epsilon 1}{2}\right)^2 \Big/ (2^2 \cdot 3^2 \cdot 7).$$

Since $2 \mid b_2$, v is odd. Thus $4 \mid (q-\epsilon 1)$. If $7 \mid (q-\epsilon 1)$, then $7 \mid b_2$, a contradiction. Hence 7 does not divide $q-\epsilon 1$. When $3 \mid (q-\epsilon 1)$, by (1) we have $b_2 = (q-\epsilon 1)^2/(2^4 \cdot 3^2)$, it follows that $q-\epsilon 1$ divides $2^3 \cdot 3^4$. Note that $2 \mid b_2$ and $2^3 \mid (q-\epsilon 1)$, we get

$$q \in \{23, 71, 73, 217, 647, 649\}.$$

Hence we can get a contradiction as in Lemma 4.2, and hence Lemma 4.4 is proved. □

Lemma 4.5. *3 does not divide b_2.*

Proof. Suppose that $3 \mid b_2$. Then by Lemmas 4.1–4.4, we can assume that $3 \nmid q$ and $b_2 = 3^i$ with $i \geq 1$. Let $3^a \parallel (q - \epsilon 1)$ $(a \geq 1)$. Then $T \cong (Z_{3^a} \times Z_{3^a}) \cdot 3$. Suppose that $Z_{3^a} \times Z_{3^a} \leq G_L$. Then $|G_L|_3 = 3^{2a}$. Since the Sylow 3-subgroups of $G_2(q_0)$ are non-Abelian, $G_L \not\cong G_2(q_0)$. Again the Sylow 3-subgroups of $L_2(8)$ are cyclic, hence $G_L \not\cong L_2(8)$. Therefore $G_L \cong (SL_2(q) \circ SL_2(q)) \cdot 2$ from Lemma 3.1. In this case, $b = q^4(q^4 + q^2 + 1)$ and $b_2 = 3$. This forces $k_1 = 1$ or 2. Thus $v = b_1 k_1 = q^4(q^4 + q^2 + 1)/3$ or $2q^4(q^4 + q^2 + 1)/3$, and so $|G_\alpha| = 3q^2(q^2 - 1)^2$ or $3q^2(q^2 - 1)^2/2$. By $|G_\alpha|$ divides $|G_C|$ and Lemma 3.1, we get $G_C \cong SU_3(5) : 2$. Since $3 \mid (v - 1)$, we get $v = 135625$ and $k_1 = 1$. This leads to $k_2^2 - k_2 - 45208 = 0$, which contradicts with k_2 is an integer. Hence $Z_{3^a} \times Z_{3^a} \not\leq G_L$. By Lemma 3.6, $N_G(Z_{3^a} \times Z_{3^a}) \leq G_\alpha$. It follows that $(Z_{q-\epsilon 1} \times Z_{q-\epsilon 1}) \cdot D_{12} \leq G_\alpha$. By the maximality of $(Z_{q-\epsilon 1} \times Z_{q-\epsilon 1}) \cdot D_{12}$ in $SL_3^\epsilon(q) : 2$, we obtain $G_\alpha \cong (Z_{q-\epsilon 1} \times Z_{q-\epsilon 1}) \cdot D_{12}$ and $G_C \cong SL_3^\epsilon(q) : 2$. Since $|G_L| < |G_\alpha|$ and $Z_{3^a} \times Z_{3^a} \not\leq G_L$, we have

$$G_L \in \{2^{3\cdot} L_3(2), PGL_2(11), L_2(8), L_2(13), G_2(2), G_2(q_0)\}.$$

If G_L is one of the groups $PGL_2(11)$, $L_2(8)$ and $L_2(13)$, then b is even. Since b_2 is odd, we get b_1 is even, and it follows that v is even and so $2 \parallel (q - \epsilon 1)$. Note that $|PGL_2(11)|_2 = 8$, $|L_2(8)|_2 = 8$ and $|L_2(13)|_2 = 4$. Thus by Lemma 3.8 and $|G_\alpha|_2 = 2^4$, we can get a contradiction.

Suppose that $G_L \cong G_2(q_0)$. In this case, $b_2 = (q - \epsilon 1)^2/(q_0 - \epsilon 1)^2$, which contradicts with Lemma 3.9.

Suppose that $G_L \cong 2^{3\cdot} L_3(2)$. Then $b_2 = (q - \epsilon 1)^2/4$. By $b_2 \mid (c - 1)$, we deduce that $q - \epsilon 1$ divides 18. Thus we have

$$q \in \{5, 7, 17, 19\}.$$

We obtain a contradiction as in Lemma 4.2.

Therefore we can assume that $G_L \cong G_2(2)$. In this case, $b_2 = (q - \epsilon 1)^2/36$. We get that $q - \epsilon 1$ divides $2 \cdot 3^4$. Hence

$$q \in \{17, 19, 53, 161, 163\}.$$

We obtain a contradiction again as in Lemma 4.2. This final contradiction shows that Lemma 4.5 is true. □

Since b_2 divides $|G|$, by Lemmas 4.1–4.5, we have $b_2 = 1$ and, consequently, $\mathcal{D}$ is a projective plane, a contradiction.

This completes the proof of Proposition 4.2. □

Remark. In the proof of Proposition 4.2, we need to check Eq. (2) has the integer number solutions or not for k. (Note that when q is given, $|G_L|$ and $|G_\alpha|$ are decided, and so (2) is an equation for k.) However, for the $q's$ occurring in Proposition 4.2, it is possible that the number in the right of (2) is not an integer. In this case, of course, (2) has no integer number solutions for k. Our checking process is completed by using the software *Mathematica*.

Now we can prove our Main Theorem stated in the Introduction.

Suppose that $\mathrm{Soc}(G) \cong G_2(q) = H_0$. Assume that G contains a graph automorphism of $G_2(q)$. Then by Proposition 4.1, we know that $G_\alpha \cap H_0$ and $G_L \cap H_0$ are groups occurring in Lemma 3.2. By line-primitivity of G, we get $b = |H_0|/|G_L \cap H_0|$ and $v = |H_0|/|G_\alpha \cap H_0|$. From [14], we know that $\mathcal{D}$ is not a projective plane. Thus $b > v$. By Lemma 3.7, for any prime t dividing b_2, there exists a Sylow t-subgroup of G such that $N_G(T) \leq G_\alpha$. In this case we prove the main theorem in several steps.

(i) If $G_\alpha \cap H_0 \cong B_0$, a Borel subgroup of G, then $v = |H_0/(G_\alpha \cap H_0)| = (q+1)^2(q^4+q^2+1)$. From $v-1 = q(q^5+2q^4+2q^3+2q^2+2q+2)$ and $b \mid v(v-1)$, we conclude that $|b|_3$ divides q and $q^5 \mid |G_L|$. By Lemma 3.2 and $|G_L| < |G_\alpha|$, there is no such G_L.

It follows that $3 \mid v$ and thus $3 \nmid b_2$. Let Q be a Sylow 3-subgroup of $G_\alpha \cap H_0$, then by Lemma 3.8, $Q \leq G_L \cap H_0$ for some $L \in \mathbf{L}$.

(ii) If $G_\alpha \cap H_0 \cong (Z_{q-\epsilon 1})^2 \cdot D_{12}$, then $v = q^6(q+\epsilon 1)^2(q^4+q^2+1)/12$. It follows that $v-1 = (q-\epsilon 1)(q^{11}+\epsilon 3q^{10}+5q^9+\epsilon 7q^8+9q^7+\epsilon 11q^6+12q^5+\epsilon 12q^4+12q^3+\epsilon 12q^2+12q+\epsilon 12)/12$. By $|G_L \cap H_0| < |G_\alpha \cap H_0|$ and Lemma 3.2, we have $G_L \cap H_0 \cong (Z_{q-1})^2 : D_{12}$ (and $\epsilon = -$) or $Z_{q^2\pm q+1} : Z_6$ or $G_2(q_0)$. If $G_L \cap H_0 \cong (Z_{q-1})^2 \cdot D_{12}$, then

$$b = |H_0|/|G_L \cap H_0| = q^6(q+1)^2(q^4+q^2+1)/12.$$

In this case, $b_1 = q^6(q^4+q^2+1)/3$ and $b_2 = b/b_1 = (q+1)^2/4$. Since $b_2 \mid (v-1)$, this forces $q = 3$. By $bk(k-1) = v(v-1)$, we get $k(k-1) = 5528$, a contradiction since k is an integer. If $G_L \cap H_0 \cong Z_{q^2\pm q+1} : Z_6$, then $b = q^6(q^4+q^2+1)(q^2-1)^2/(6(q^2 \pm q+1))$ and $b_2 = b/(b,v) = 2(q-\epsilon 1)^2$. Since $q = 3^n$, it follows that $b_2 \nmid (v-1)$, a contradiction. If $G_L \cap H_0 \cong G_2(q_0)$, then $m \geq 7$. It is easy to get that $b_2 = (q-\epsilon 1)^2/(q_0-\epsilon 1)^2$. This is impossible since $b_2 \mid (v-1)$.

(iii) If $G_\alpha \cap H_0 \cong (SL_2(q) \circ SL_2(q)) \cdot 2$, then by $Q \leq G_L$ and $|G_L| < |G_\alpha|$ and Lemma 3.2, there does not exist such G_L.

(iv) If $G_\alpha \cap H_0 \cong 2^{3\cdot}L_3(2)$, then $q = 3$ and $v = 3^5 \cdot 13$. By $|G_L \cap H_0| < |G_\alpha \cap H_0|$ and $q = 3$, we get $G_L \cap H_0 \cong L_2(13)$ or $(SL_2(3) \circ SL_2(3)) \cdot 2$, and so $b = 2^4 \cdot 3^5$ or $3^4 \cdot 7 \cdot 13$. In both cases, b does not divide $v(v-1)$.

(v) If $G_\alpha \cap H_0 \cong G_2(q_0)$, where $q = q_0^m$ and m a prime, then $|Q| = q_0^5$. By Lemma 3.2, there is no G_L satisfying $Q \leq G_L$ and $|G_L| < |G_\alpha|$.

(vi) If $G_\alpha \cap H_0 \cong L_2(13)$, then $q = 3$. There are no G_L in Lemma 3.2 satisfying $|G_L| < |G_\alpha|$.

(vii) If $G_\alpha \cap H_0 \cong {}^2G_2(q)$, then $|Q| = q^3$. Thus by $Q \leq G_L$ and $|G_L| < |G_\alpha|$, we have $G_L \cong G_2(q_0)$ with $q = q_0^2$. In this case, $b = q^3(q^3+1)(q+1)$ and $v = q^3(q^3-1)(q+1)$. However, b does not divide $v(v-1)$.

Hence when G contains a graph automorphism of $G_2(q)$, we proved the Main Theorem is true.

Now we suppose that G does not contain any graph automorphism of $G_2(q)$. Since $G_2(q) \trianglelefteq G$ and G is line-primitive, $G_2(q)$ is line-transitive. By Lemma 3.1, $G_L \cap G_2(q)$ is a maximal subgroup of $G_2(q)$ or $G_L \cap G_2(q) \supseteq G_2(q)$. Since G is line-transitive, $G_L \cap G_2(q) \not\supseteq G_2(q)$. Thus we can assume that $G = G_2(q)$. By Proposition 4.2, we know that G_α is maximal in $G_2(q)$.

Now we prove the Main Theorem in several steps.

(i) If $G_\alpha \cong P_a$ or P_b, parabolic subgroups of $G_2(q)$, then $v = |G/G_\alpha| = (q^3+1)(q^2+q+1)$. From $v-1 = q(q^4+q^3+q^2+q+1)$ and $b \mid v(v-1)$, we deduce that $|b|_3$ divides q and $q^5 \mid |G_L|$. By Lemma 3.1 and $|G_L| < |G_\alpha|$, there is no such G_L.

Thus G_α does not contain a Sylow 3-subgroup of G. Let Q be a Sylow p-subgroup of G_α, then by Lemma 3.8, $Q \le G_L$ for some $L \in \mathbf{L}$.

(ii) If $G_\alpha \cong SL_3^\epsilon(q) : 2$, then by $Q \le G_L$ and $|G_L| < |G_\alpha|$ and Lemma 3.1, we have $G_L \cong SL_3(q) : 2$ and $\epsilon = -$. In this case, $b = q^3(q^3+1)/2$ and $v = q^3(q^3-1)/2$ and $v-1 = (q^3-2)(q^3+1)/2$. But b does not divide $v(v-1)$.

(iii) If $G_\alpha \cong (SL_2(q) \circ SL_2(q)) \cdot 2$, then by $Q \le G_L$ and $|G_L| < |G_\alpha|$ and Lemma 3.1, there does not exist such G_L.

(iv) If $G_\alpha \cong G_2(q_0)$, where $q = q_0^m$ and m is a prime, then $|Q| = q_0^6$. By Lemma 3.1, there is no G_L satisfying $Q \le G_L$ and $|G_L| < |G_\alpha|$.

(v) If $G_\alpha \cong L_2(13)$, then by $|G_L| < |G_\alpha|$, we get $G_L \cong L_2(8)$. But both $G_L \cong L_2(8)$ and $L_2(13)$ cannot be maximal in G at the same time (refer to the remarks in Lemma 3.1). Thus $G_\alpha \not\cong L_2(13)$.

(vi) If $G_\alpha \cong 2^{3\cdot}L_3(2)$, then $q = p$. By $|G_L| < |G_\alpha|$, we get $G_L \cong L_2(8)$ or $L_2(13)$. But when $G_L \cong L_2(8)$ or $L_2(13)$ is a maximal subgroup of G, $q \ne p$, a contradiction. Thus $G_\alpha \not\cong 2^{3\cdot}L_3(2)$.

(vii) If $G_\alpha \cong {}^2G_2(q)$, then $v = q^3(q^3-1)(q+1)$ and $v-1 = q^7+q^6-q^4-q^3-1$. In this case, $|Q| = q^3$. By $Q \le G_L$ and $|G_L| < |G_\alpha|$, we get $G_L \cong G_2(q_0)$, where $q = q_0^2$. Therefore, $b = |G : G_L| = q^3(q^3+1)(q+1)$. It follows that $(q+1)^2$ divides b but not v (note that $q = 3^n$). This is impossible.

(viii) $G_\alpha \not\cong L_2(8)$ since the order of $L_2(8)$ is the least of the groups in Lemma 3.1.

(ix) If $G_\alpha \cong G_2(2)$, then $q = p$. Thus $G_L \cong 2^{3\cdot}L_3(2)$ or $PGL_2(q)$ and $q \in \{11, 13, 17, 19\}$. If $G_L \cong PGL_2(q)$, then $q \in \{11, 13, 17, 19\}$. When $q = 17$, v is even, and when $q = 19$, $3 \mid v$. Both cases conflict with the following equation

$$k(k-1) = |G_L|(v-1)/|G_\alpha|.$$

When $q = 11$, $v-1 = 2 \cdot 3 \cdot 13 \cdot 13903 \cdot 28711$. When $q = 13$, $v-1 = 2 \cdot 3 \cdot 5^2 \cdot 11 \cdot 19 \cdot 97^2 \cdot 1097$. In two cases, $2 \parallel (v-1)$. This conflicts with the equation

$$k(k-1) = |G_L|(v-1)/|G_\alpha|.$$

If $G_L \cong 2^{3\cdot}L_3(2)$, then $b = 9v$. Thus $b_2 = 9$ and $3 \nmid v$. It follows that $|G|_3 = 27$. Let $3 \mid (q-\epsilon 1)$. Then $3 \parallel (q-\epsilon 1)$. Since $Z_3 \times Z_3 \not\le G_L$, we have from Lemma 3.6 $N_G(Z_3 \times Z_3) \le G_\alpha$. Note that

$$Z_3 \times Z_3 \quad \text{char} \quad Z_{q-\epsilon 1} \times Z_{q-\epsilon 1} \trianglelefteq (Z_{q-\epsilon 1} \times Z_{q-\epsilon 1}) \cdot D_{12},$$

we have $(Z_{q-\epsilon 1} \times Z_{q-\epsilon 1}) \cdot D_{12} \le G_\alpha$, a contradiction.

(x) If $G_\alpha \cong J_1$, then $q = 11$. Thus $11 \mid |G_L|$ by Lemma 3.8. By $|G_L| < |G_\alpha|$, $G_L \cong PGL_2(11)$. Since v is even, the equality

$$k(k-1) = |G_L|(v-1)/|G_\alpha|$$

cannot hold.

(xi) If $G_\alpha \cong PGL_2(q)$, then $p \geq 7$ and $q \geq 11$ and $q \mid |G_L|$. By $|G_L| < |G_\alpha|$, we get $G_L \cong G_2(q_0)$ and $q = q_0^5$. In this case, $v = q_0^{25}(q_0^{30} - 1)$. By Lemma 3.4, there exists a prime u, such that $u \mid (q_0^{10} - 1)$ and $u \nmid (q_0^6 - 1)$. Since

$$k(k-1) = |G_L|(v-1)/|G_\alpha| = q_0(q_0^2 - 1)(q_0^6 - 1)(v-1)/(q_0^{10} - 1),$$

we know that $u \mid (v - 1)$, which conflicts with $u \mid v$.

Hence when G does not contain any graph automorphism of $G_2(q)$, we also proved the Main Theorem is true.

Now we have completed the proof of the Main Theorem.

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